Multifractality of time and space, covariant derivatives and gauge invariance

L.Ya.Kobelev
Department of Physics, Urals State University
Lenina Ave., 51, Ekaterinburg 620083, Russia
E-mail: leonid.kobelev@usu.ru

A possibility to represent the standard model of fundamental particles covariant derivatives by means of approximate generaleralized fractional Riemann-Liouville derivatives of multifractal time and space model is shown.

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I. INTRODUCTION

1. In the Riemannian Geometry and in the standard theory of fundamental particles (see for example, [1], [2])the covariant derivatives are used, instead of usual derivatives. In the standard theory of fundamental particles the covariant derivatives has the form

$$D^\mu = \partial^\mu - ig_1 \frac{Y}{2} B^\mu - ig_2 \frac{\tau_i}{2} W_i^\mu - ig_3 \frac{\lambda_a}{2} G_a^\mu \qquad (1)$$
$$i = 1,2,3, \quad a = 1,2,..8$$

where B^μ, W_i^μ, G_a^μ are gauge-invariant vector fields with symmetry groups $U(1)$,$SU(2)$,$SU(3)$, τ_i,λ_a - are matrices of isospin and color fields, Y - hypercharge, $\mu = 0,1,2,3$. The covariant derivatives with respect to tensor $t^\mu\nu$, in space with a metric tensor $\gamma^\mu\nu$, are defined by relations

$$D_\alpha t^{\mu\nu} = \partial_\alpha t^{\mu\nu} + \gamma^\nu_{\alpha\beta} t^{\mu\beta} \qquad (2)$$

where $\gamma^\nu_{\alpha\beta}$ are Kristoffel's coefficients

$$\gamma^\nu_{\alpha\beta} = \frac{1}{2}\gamma^{\mu\sigma}(\partial_\alpha \gamma_{\beta\sigma} + \partial_\beta \gamma_{\alpha\sigma} + \partial_\sigma \gamma_{\alpha\beta}) \qquad (3)$$

In this and other cases to ∂^μ adds vector, tensor etc. functional terms usual derivatives (in last cases ∂^μ is multiplied on a unit matrix of the according type).

2. For describing of dynamic characteristics of systems defined on multifractal time and space sets it is necessary to define the functionals (left-sided and right-sided) determined on the functions, given on a multifractal sets. These functionals are the elementary generalization of the Riemann - Liouville fractional derivatives and integrals (about the Riemann-Liouville fractional derivatives see [3]) for functions defined on multifractal sets and were introduced in [4]. These functionals (generalized fractional derivatives (GFD)) read (for differentiation with respect to time)

$$D^d_{+,t} f(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(t')dt'}{\Gamma(n-d(t'))(t-t')^{d(t')-n+1}} \qquad (4)$$

$$D^d_{-,t} f(t) =$$
$$= (-1)^n \left(\frac{d}{dt}\right)^n \int_t^b \frac{f(t')dt'}{\Gamma(n-d(t'))(t'-t)^{d(t')-n+1}} \qquad (5)$$

where Γ-is a gamma function, $a < b$, a and b stationary values selected on an infinite axis (from $-\infty$ to ∞), $n-1 \le d_t < n$, $n = \{d_t\}+1$, $\{d_t\}$- integer part of $d_t \ge 0$ and $n = 0$ for $d_t < 0$. Generalized fractional derivatives (GFD) (4)-(5) coincide with fractional derivatives or fractional integrals of the Riemann - Liouville for a case $d_t = const.$ At $d_t = n + \varepsilon(t)$, $\varepsilon(t) \to 0$ GFD are represented through usual derivatives and integrals [4]. The functions in integrals in (4)-(5) are generalized functions, given on set of a finitary functions [5]. Similar (4)-(5) definitions GFD can be defined and with respect to space variables $\mathbf{r}$. The integral functionals (4)-(5) allow to describe dynamic of functions defined on multifractal sets and replace for such functions orderly or fractional (Riemann-Liouville) differentiation and integration. These functionals partially describe the memory about past (or future, if use right-hand GFD) time or spatial events. In [4] it is shown, for $d_\alpha = 1 * \varepsilon_\alpha(\mathbf{r}(t), t)$, $\alpha = t, \mathbf{r}$, $|\varepsilon| \ll 1$ where

$$\varepsilon_\alpha = \sum_i \beta_{\alpha,i} L_{\alpha,i}(\Phi_i) \qquad (6)$$

($L_{t,i}$ -are the Lagrangians densities of energy of physical fields which presents in a point x, $\beta_{t,i}$ are a numerical dimension factor, $L_{\mathbf{r},i}$- are the Lagrangians densities of energy of the new fields which appears be-cause of fractionality of space dimensions [4]) that GFD can be represented in that case by usual derivatives (scalar field):

$$D^{1+\varepsilon_\alpha(x)}_{+,x^\mu} f(x) = \partial^\mu f(x) + a\partial^\mu[\varepsilon_\alpha(x) f(x)] \qquad (7)$$

where a -is a factor of the regularization of integrals (4)-(5) or an analogies integrals for case of differentiation with respect to space coordinates. The relations (6), defining GFD for sets with almost integer dimension, have more composite structure than (1) and (2), nevertheless, they are very similar to definitions of the covariant derivatives. For a gravitational field the connections GFD (for the component $\Phi_0 0$ of a gravity's potentials)

with covariant derivatives of effective Riemannian space was obtained in [4]. The purpose of this paper is the establishment of the connections between GFD defined by (4)-(6) and covariant derivatives (1)-(2), for taking into account only the feeble physical fields and multifractal nature of time and using the relations for covariant derivatives (4)-(6) which are more complicate, than derivatives defined by relations (1) - (2).

II. COVARIANT DERIVATIVES AND MULTIFRACTAL TIME

1. We shall be restricted by consideration of connection GFD and covariant derivatives defined on the multifractal time sets (i.e. $\alpha = t$, for multifractal 3-dimensions coordinates space the consideration may be carried out by similarly methods) for a case of the small fractional corrections to integer dimension of $t(|\varepsilon| \ll 1)$. From (7) follows

$$D^{1+\varepsilon(x)}_{+,x^\mu} f(x) = [(1+\varepsilon)\partial^\mu - \partial^\mu \varepsilon(x)] f(x) \tag{8}$$

or, with the account of (6)

$$D^{1+\varepsilon(x)}_{+,x^\mu} f(x) = [1 - \sum_i \beta_i L_i(\Phi_i(\mathbf{r}(t),t))]\partial^\mu f(x) - \sum_i \beta_i \partial^\mu [L_i(\Phi_i(\mathbf{r}(t),t)) f(x)] \tag{9}$$

The relations (9) defines, for $|\varepsilon| \ll 1$, connections between GFD (in the set of multifractal time with dimension $d = 1+\varepsilon$) and orderly derivatives with respect to time and coordinates in the time's set with dimension $d = 1$. Note that from the point of view of the multifractal theory, it is possible to treat the time's set with dimension $d = 1$, in which derivatives with respect to time and coordinates are replaced on covariant derivatives of the (9), as the effective time's set corresponding to multifractal time with dimension $d = 1+\varepsilon$.

2. The comparison (9) and (1) allows to establish difference of general structure of derivatives (9) from covariant derivatives (1): except for presence more composite, than in (1), additive terms proportional derivatives of Lagrangians density with respect to time and coordinates, there are renormalization of the function's factors before derivatives ∂^μ, which quantity depends both on time and from coordinates. The presence this renormalization allows to introduce, instead of space of time with fractional dimension for tensor fields an effective Riemannian space with the metric defined by dependence of ε from a metric tensor (see special case in [4]).

3. What is the physical sense of replacement of usual derivatives on the covariant derivatives (9)? The derivatives of the Lagrangians density with respect to time and coordinates can be interpreted as a birth or disappearance of energy (the signs of derivatives are plus or are minus, accordingly) if the time or the coordinates are changing. This energy is transmitted to a field of time or is taken from it by the carrier of a measure (by the set R^n). The conservation laws are fulfilled for closed system consisting of material fields of the time and the space and set of the carrier of a measure R^n. The fields of time and space without the carrier of measure set are open systems. The birth or annihilation of energy of the field of time is accompanied by changes of before derivatives quantity (factor before derivatives ∂^μ) depending not only from characteristics of function $f(t)$, with respect to which the operation of differentiation is applied, but also from characteristics of the field of time (defined by Lagrangians density in the given instant and given coordinates). The comparison (9) and (1) allows to state if the selection of a Lagrangians in the form of standard model of the theory of fundamental particles (with replacement in a Lagrangians the usual derivatives by covariant derivatives (1)) is made, that the covariant derivatives (9) contain the description of process of birth or disappearance all physical fields (for example electromagnetic fields or fields of fundamental particles). The intensity of these processes depends on density of energy and, thus (9) are not reduced to (1) at the any selections of Lagrangians of standard model. The comparison (9) with the covariant derivatives general theory of relativity (GTR) (2) allows to state, as the components of a metric tensor are functions of an energy-momentum tensor of gravitational field $t_{\mu\nu}$, that covariant derivatives (9) contains (2) as a special case (according to presence only one gravitational field). This case was considered in [4] and it was demonstrated the opportunity of introduction of effective Riemannian space (according to Riemannian space of GFR) for approximate describing of gravitational phenomenon in multi-fractal time (if the time is multifractal in reality) at small ε.

III. GAUGE INVARIANCE OF THE FIELDS $\partial_\mu \sum_I \beta_I L_I$

Are the fields $\partial^\mu \sum_i \beta_i L_i$ gouge invariant? Whether is it necessary to require the gauge invariance of GFD (for small ε) from the equations of theoretical physics which are wrote down with the help of GFD for multifractal sets, where time and space have fractional dimensions almost undistinguished from the integer dimensions (these equations are wrote down in [4])? In the last case if to follow fields quantum theory, the requirement of the gauge invariance introduce the new "charges", that is the charges, defining birth or disappearance of fields L_i. This charge, is defined by both factors β_i and factors included in Lagrangians (that define already known charges). The requirement of the gouge invariance enters as well the new massless physical fields: the field's of production of all known physical fields φ:

$$\varphi^\mu(\mathbf{r}(t),t), L_i) = \partial^\mu \sum \beta_i L_i \tag{10}$$

As the relations (9) for GFD are approximate and are valid only for small ε, the gauge invariance (10) is also approximate and it is meaningful (if it at all is meaningful) only for small ε (though this case is most spread). The problem of validity of introduction of a gauge invariance (invariance of derivatives densities of Lagrangians) remains open.

IV. IS THE FRACTAL DIMENSION OF TIME INFLUENCED BY CHARACTERISTICS OF MULTIFRACTAL SETS THAT ARISE IN THE INTERNAL SETS OF "TIME INTERVALS"?

The form of the covariant derivatives in the standard theories of fundamental particles (1), allow (within the framework of representations of multifractal time) to put the problem: is it possible to construct dimension of time d_t in such a manner that covariant derivatives (1) will be appears in the theory by explicit form, instead of appears through Lagrangians? As will be shown below, it is possible, being non-essential for d_t. It is possible presents the covariant derivatives (9) in the form more similar to covariant derivatives (1), including the relations (1) as a special case. Till now "time intervals" from which, on definition, the material field of time is consists, were treated as "points" of time sets. It was considered possible to neglect by interior structure of time sets component those "time intervals". The covariant derivatives (1) describe characteristics of interior symmetry of fundamental particles and their gauge invariance, so, apparently, it is convenient in multifractal model investigate the characteristics of inner sets that consist the time and the space "points". So let's add to density of Lagrangians (in ε that defined the multifractal dimensions of time and space sets) the terms, which origin can be connected with characteristics of sets constructing the "points" by the vicinity of points x. These time and space sets (intervals) near x earlier approximately were described as the "points" and characterized by fractional dimension d_t or d_r, but that part of sets contents and carry the information about structural characteristics of fundamental particles (as the field of time, following [4], generates all material fields and defines their characteristics). If the sets components of "time intervals" are multifractal, than to each point this "interior" sets surrounding the medial "point" with coordinate x should be compared with the fractal dimensions or with their medial integral description. Most simply in this case for $d = 1+\varepsilon(\mathbf{r}(t), t)$ to write down d as

$$d = 1+\varepsilon = 1+\sum_i \beta_i L_{0,i}(\Phi_i) + \sum_{i,\mu} \int_{x^\mu}^{x_0^\mu} \tilde{B}_i^\mu(x) dx^\mu \quad (11)$$

where $\tilde{B}^\mu$ - are vector quantifies (possessing by complicated interior symmetry) and define characteristics of sets in a vicinity of each point x of the time (or space) intervals, in which this point is considered. In particular, for definition of dependence $\tilde{B}^\mu$ from physical fields and interior symmetries of fundamental particles, for example, relations (1) can be chosen. In (11) x_0^μ - are stationary values proportional to "size" of the according time intervals (or space intervals), $L_{0,i}$ -are densities of Lagrangians of free fields. As x_0^μ are very small, their contributions are essential only for derivatives $D^{1+\varepsilon}_{\pm,x^\mu}$, which in this case will accept the form

$$D^{1+\varepsilon(x)}_{+,x^\mu} \approx [1-\sum_i \beta_i L_{0,i}(\Phi_i(\mathbf{r}(t),t))]\partial^\mu - \\ -\sum_i \beta_i \partial^\mu L_{0,i}(\Phi_i(\mathbf{r}(t),t)) - \sum_i B_i^\mu(\mathbf{r}(t),t) \quad (12)$$

It is possible to rewrite covariant derivatives (12) as

$$D^{1+\varepsilon(x)}_{+,x^\mu} \approx [1-\sum_i \beta_i L_{0,i}(\Phi_i(\mathbf{r}(t),t))] \times \\ \times \partial^\mu - \frac{\sum_i [\beta_i \partial^\mu L_{0,i}(\Phi_i(\mathbf{r}(t),t)) - B_i^\mu(\mathbf{r}(t),t)]}{[1-\sum_i \beta_i L_{0,i}(\Phi_i(\mathbf{r}(t),t))]} \quad (13)$$

If B_i^μ in (12) determine by use of relations (1) and if neglect by the contributions from fractal dimensions in square brackets ($\beta \to 0$), the definition covariant derivatives (12) coincides with the covariant derivatives of standard theory of fundamental particles (at the according selection of stationary values and matrices). It is necessary to note, apparently, the equivalence of mathematical expositions of dynamic characteristics of fundamental particles in viewed model as with the help of use of densities of Lagrangians (into which the terms describing interactions of fields and particles are included), and exposition with the help of the introduction of covariant derivatives. In the last case and elimination, from the according densities of Lagrangians of terms describing interactions must be made.

V. CONCLUSIONS

In the presented theory of multifractal time and space the generalized fractional derivatives, defined by (4) - (5) are used instead of usual differentiation and integration for describing a dynamic characteristics of any physical objects (fields, particles etc. The using of integral functionals gives in considerable thickening of the mathematical tool and change a great deal representations about the physical nature of time and space. Only in the case when the fractal corrections to FD are small, it is possible to present GFD with the help of usual derivatives and integrals as was shown in [4] and used in this paper. In this case the fractionality of dimensions of time (or, similarly, dimensions of space, see [4]) can be presented by introduction of an "effective" time and space, in which derivatives (and the integrals, for a case of

negative fractional dimensions) are substituted by "covariant" derivatives and integrals. The consideration of concrete models and Lagrangians, in particular, the Lagrangians of the standard theory of fundamental particles, has been illustrated an opportunity of exposition of gouge invariant fields with covariant derivatives of a new type (12). This covariant derivatives are contains, except for known terms of the gouge invariant standard theory of fundamental particles and the terms including of effective Riemannian space, the terms describing the continuous change of densities of energy of physical fields (their birth or disappearance). Multifractal set of time used in the present paper (also, as well as the mathematical tool GFD, used in a series of other papers [6] - [8]) creates peculiar model of the world considered as the open system (see [9], [10]), in which there are no invariable states. In viewed model of multifractal time all physical fields are not stationary (note that GFD with respect to time or coordinates of stationary values are not equal to zero), but also their energy continuously changing thow potentials are invariable ("non-potential" change of energy as the result of an exchange of energy with the set of the carrier of a measure, because of the presence at the covariant derivatives the terms with derivatives from densities of Lagrangians. At least, for the case $|\varepsilon| \ll 1$, the theory of fundamental particles defined on sets of multifractal time and space, allows to take into account by the uniform mathematical method the influences of all known physical fields (the fundamental particles, used by the theory, and gravitational field). It is achieved by introduction the new paradigm [4]: the exposition of time and space as multifractal sets with fractional dimensions and introduction, in this connection, mathematical methods of generalized fractional derivatives and integrals. For deriving the modified equations of the standard theory of fundamental particles or various variants of the great unification theory (or any physical theory) it is enough to describe the fields taking into account the multifractal nature of time and space, i.e. replace covariant derivatives such as (1) (or an orderly derivatives in a scalar theories) by generalized covariant derivatives (GFD) (12) or, depending on selection of models, by similar though more composite covariant derivatives for fields with more complicated mathematical nature. The problems of correspondence obtained by such replacement models of the theories of fundamental particles to characteristics of the real world (if use model of multifractal time and spaces presented in [4]) here are not considered. All equations of theoretical physics wrote down with the help the GFD for the case $\varepsilon = 0$ can be considered as special cases of equations of presented model of multifractal time and space . At last we pay attention on the appearance of the new characteristics in modified thus theories (which were not explored yet) stipulated by the additional factors, and by the new terms in the modified covariant derivatives.

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